Orbit Stabilizer Theorem and Its Applications to Conjugacy and p-Groups

Onyie Okoye

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Introduction

Group actions provide a useful way to study groups through the symmetries they create on sets. Many important ideas in algebra, such as stabilizers and orbits, arise naturally when a group acts on some object of interest. One central result about actions is the Orbit Stabilizer Theorem. It connects the structure of the orbit of an element with the size of the subgroup that fixes that element. This relationship appears in many settings, and it often allows us to understand the size and structure of a group using the way it acts.

In this paper I will state and prove a version of the Orbit Stabilizer Theorem that describes the action on a transitive set in terms of an action on a coset space. After this, I will apply the theorem to the conjugation action of a group on itself. This will give information about the sizes of conjugacy classes and centralizers. As a final application, I will use these facts to prove a basic property of p groups.

Definitions

Let G be a group and let X be a set. An action of G on X is a function $G \times X \to X$ written $(g, x) \mapsto g \cdot x$ with the properties that $e \cdot x = x$ for all $x \in X$ and $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

As a simple example, the symmetric group S_3 acts on the set $\{1, 2, 3\}$ by permuting the elements. In this case $g \cdot x$ is the usual application of the permutation g to the number x.

For $x \in X$, the orbit of x is the set

$$\text{Orb}(x) = \{g \cdot x \mid g \in G\}.$$

In the example above, the orbit of 1 under the action of S_3 is the entire set $\{1, 2, 3\}$ since there is a permutation sending 1 to each of the other elements.

The stabilizer of x is the subgroup

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

For instance, the stabilizer of 1 in the S_3 action consists of the identity and the transposition (2 3).

The action of G on X is called transitive if for any $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$. In this situation, the entire set X is a single orbit.

If G acts on sets X and Y, the actions are said to be equivalent if there exists a bijection $\phi : X \to Y$ such that for all $x \in X$ and all $g \in G$,

$$\phi(g \cdot x) = g \cdot \phi(x).$$

If H is a subgroup of G, the set G/H denotes the set of left cosets gH. The action of G on G/H is given by $g \cdot (xH) = gxH$.

As an example, let $G = \mathbb{Z}_6$ acting on the cosets of the subgroup $H = \{0, 3\}$. The cosets are $0 + H = \{0, 3\}$, $1 + H = \{1, 4\}$, and $2 + H = \{2, 5\}$. Addition in $\mathbb{Z}_6$ moves one coset to another, so this is a transitive action.

The Orbit Stabilizer Theorem

Theorem 1 (Orbit Stabilizer Theorem) *Let G be a finite group acting transitively on a set X. Let $a \in X$, and let H be the stabilizer of a. Then the action of G on X is equivalent to the action of G on the coset space G/H.*

Since the action is transitive, every element of X has the form $g \cdot a$ for some $g \in G$. This suggests that we should relate $g \cdot a$ to the coset gH. Define a map $\phi : X \to G/H$ by

$$\phi(g \cdot a) = gH.$$

To see that this is well defined, suppose $g \cdot a = g' \cdot a$. Then $g^{-1}g'$ stabilizes a, so $g^{-1}g' \in H$. This is equivalent to $gH = g'H$. Thus the value of $\phi(g \cdot a)$ does not depend on the choice of g.

The map is surjective because every coset has the form gH and is equal to $\phi(g \cdot a)$. It is injective because if $gH = g'H$ then $g^{-1}g' \in H$ implies $g \cdot a = g' \cdot a$.

To show that ϕ is equivariant, let $h \in G$ and $x = g \cdot a$. Then

$$\phi(h \cdot x) = \phi(hg \cdot a) = hgH = h \cdot (gH) = h \cdot \phi(x).$$

This completes the proof.

Applications to Conjugation

A group can act on itself by conjugation. For $g, x \in G$, the action is defined by

$$g \cdot x = gxg^{-1}.$$

This action describes how elements move under internal symmetries. For example, in S_3, conjugating the transposition $(1\,2)$ by a permutation g produces the transposition $(g(1)\,g(2))$. Thus conjugate elements behave the same way inside the group.

The orbit of x under this action is its conjugacy class, and the stabilizer of x is the centralizer

$$C_G(x) = \{g \in G \mid gxg^{-1} = x\}.$$

Proposition 2 *Let G be a finite group and let $x \in G$ be an element of order k. Then k divides $|G|$, and the number of conjugates of x in G divides $|G|/k$.*

The element x generates a cyclic subgroup $\langle x \rangle$ of order k. Since every element of $\langle x \rangle$ commutes with x, the subgroup is contained in the centralizer $C_G(x)$. Thus k divides $|C_G(x)|$.

Under conjugation, the orbit of x is its conjugacy class, and the stabilizer is $C_G(x)$. The Orbit Stabilizer Theorem gives

$$|G| = |\operatorname{Orb}(x)| \cdot |C_G(x)|.$$

Since k divides $|C_G(x)|$, it follows that k divides $|G|$.

The number of conjugates of x is $|\operatorname{Orb}(x)|$, and from the equation above

$$|\operatorname{Orb}(x)| = \frac{|G|}{|C_G(x)|}.$$

Since $|C_G(x)|$ is a multiple of k, the orbit size divides $|G|/k$.

Application to p Groups

Proposition 3 *Assume that $|G| = p^k$ for some $k > 0$, and let Z be the center of G. Then $|Z|$ is a multiple of p.*

Partition the group into its conjugacy classes. Central elements form classes of size 1. If x is not central, its conjugacy class has more than one element.

By the previous proposition, the size of the conjugacy class of a noncentral element divides $|G| = p^k$, and since it is greater than 1 it must be a positive power of p. Hence every such class has size divisible by p.

Let the conjugacy classes be $C_1, \dots, C_m$, with $C_1, \dots, C_r$ the central ones. The class equation is

$$|G| = |C_1| + \cdots + |C_r| + |C_{r+1}| + \cdots + |C_m|.$$

All noncentral classes have sizes divisible by p, so subtracting their sum leaves

$$|C_1| + \cdots + |C_r| = |Z|.$$

This is a multiple of p because $|G|$ is a multiple of p and the other terms are multiples of p.

Conclusion

The Orbit Stabilizer Theorem provides a simple relationship between the size of an orbit and the size of a stabilizer, but it also describes how a transitive action can be viewed as an action on a coset space. This viewpoint turns out to be very useful when studying the conjugation action of a group on itself. Conjugacy classes reveal how elements fail to commute, and their sizes give information about centralizers and the structure of the group. The class equation combines these observations and shows that p groups must have nontrivial centers. These ideas illustrate how group actions connect the internal structure of a group with the way it moves other objects, and they provide tools that appear throughout the study of finite groups.